Chapter 7 Note Package

Monday, March 28, 2022 2:08 PM

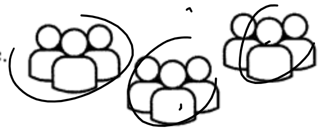
Chapter 7: Exponential Functions

7.1 Characteristics of Exponential Functions

An exponential function is a function where the ***exponent*** includes a variable, and the ***base*** is larger than zero, not equal to 1. Exponential functions are used to model many real-life situations of change – such as population growth, radioactive decay and compound interest.

$3^0 = 1$

For example –
Suppose you greet three people.  $3^1 = 3$

Each person you greeted goes on to greet 3 different people.  $3^2 = 9$

If this pattern continues, you can see that the number of people greeted grows very quickly. $3^3 = 27$

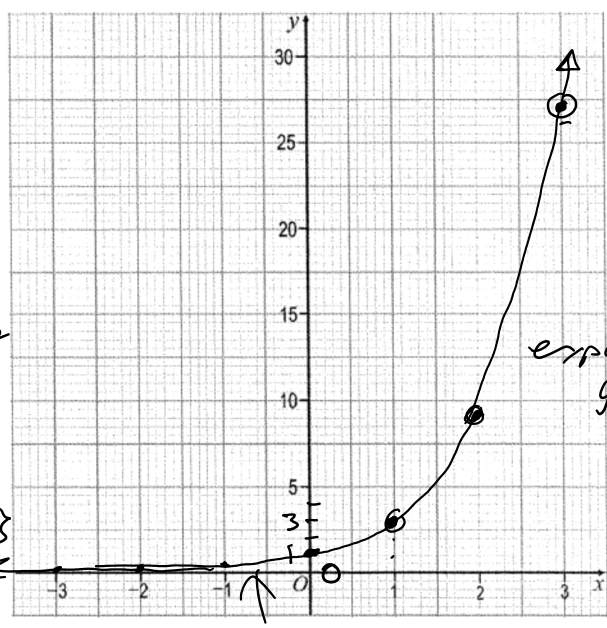
Consider the function $y = 3^x$.
a) Complete the table, then sketch the graph of $y = 3^x$ on the grid.

$y = 3^x$

x	y
0	1
1	3
2	9
3	27
-1	$\frac{1}{3}$
-2	$\frac{1}{9}$
-3	$\frac{1}{27}$

$3^{-1} = \frac{1}{3}$

$3^{-2} = \frac{1}{3^2}$



exponential growth

$\uparrow$ 2 sq = 1

b) State the graph's:
domain $\{x \mid x \in \mathbb{R}\}$
range $\{y \mid y > 0, y \in \mathbb{R}\}$
y-intercept $y = 1$
x-intercept none
horizontal asymptote equation
$y = 0$

The graph is asymptotic to the ~~x~~ axis.

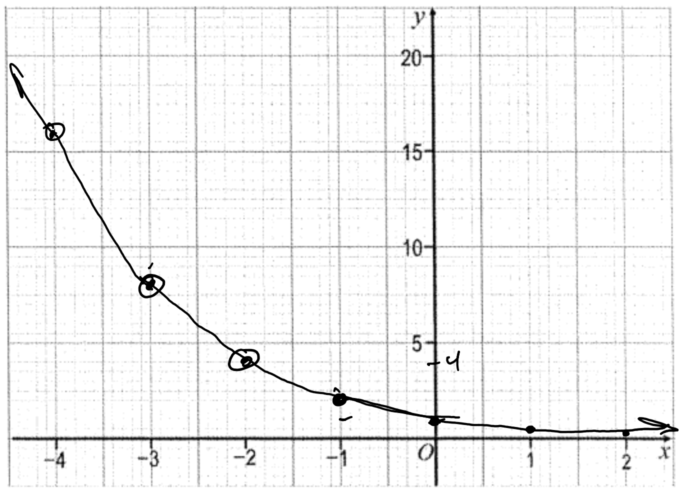
equation Asymptote: $y = 0$

Example

a) Create a table, then sketch the graph of the exponential function $y = \left(\frac{1}{2}\right)^x$ on the grid.

x	y
2	1/4
1	1/2
0	1
-1	2
-2	4
-3	8
-4	16

$y = \left(\frac{1}{2}\right)^2$
$y = \left(\frac{1}{2}\right)^1$
$y = \left(\frac{1}{2}\right)^0 = 1$
$y = \left(\frac{1}{2}\right)^{-1}$



$\left(\frac{1}{2}\right)^{-1}$
$\left(\frac{2}{1}\right)^1 = 2$
$\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 4$

exponential decay.

b) State the graph's:

domain $\{x \mid x \in \mathbb{R}\}$ range $\{y \mid y > 0\}$

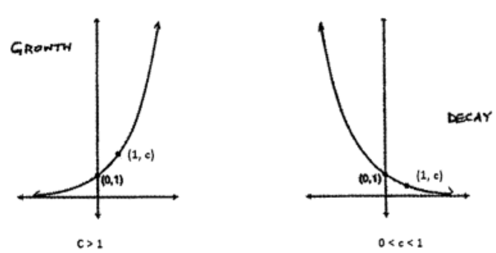
y-intercept $y = 1$ x-intercept none

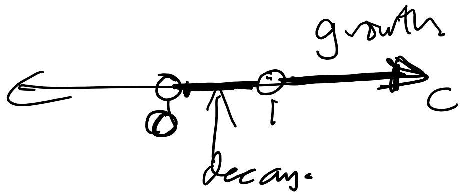
horizontal asymptote equation $y = 0$
(x-axis).

$y = c^x$, when $c > 1$ $C \neq 0$ $C \neq 1$

$y = c^x$, when $0 < c < 1$ i.e.

base $= c$. c is a fraction





7.2 Transformations of Exponential Functions

Predict what will happen to the graph of $y = 5^x$ when each of the following changes is made to the equation:

$c=5 \quad y=5^x$

- $y = 5^x + 1$ — up 1 unit.
- x replaced w/ x−4: $y = 5^{x-4}$ — 4 units right
- $\frac{1}{b} = \left(\frac{1}{2}\right)$: $y = 5^{2x}$ — HC by factor $\frac{1}{2}$
- $y = 5^{3x-12}$, $y = 5^{3(x-4)}$ — HC by factor $\frac{1}{3}$, 4 units to right.
- $y = -2(5^x)$, a — VE by factor 2, reflection over x-axis.

$y = c^x$

|a|>1 VE.
0<|a|<1 HC

Horizontal stretch by a factor of $\frac{1}{b}$

Horizontal translation

Vertical stretch by a factor of a

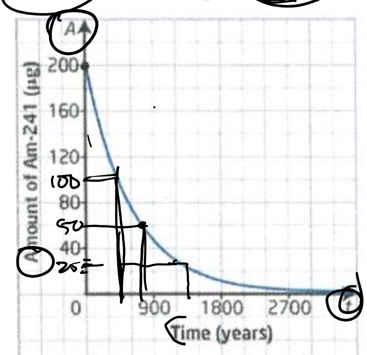
Vertical translation

$$y = a(c)^{b(x-h)} + k$$

If b < 0 then there is a reflection over the y-axis (horizontal reflection)
If a < 0 then there is a reflection over the x-axis (vertical reflection)

Creating Exponential Functions

The radioactive element americium is used in household smoke detectors. Am-241 has a half-life of approximately 432 years. The average smoke detector contains 200 μg of Am-241. What is the exponential function that models the graph showing the radioactive decay of 200 μg of Am-241? (TB, p 353)



$y = \left(\frac{1}{2}\right)^x$

$1\,\mu g = 10^{-6}$

x	y
−2	4
−1	2
0	1 ×200
1	$\frac{1}{2}$ ×200
2	$\frac{1}{4}$

x	y
0	200
432	100
864	50

VE by factor 200.
HE by 432

hor. stretch factor
$= \frac{1}{b} = 432$
$b = \frac{1}{432}$

x	y
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

x	y
0	200
432	100
864	50

$y = a\left(\frac{1}{2}\right)^{bx}$

$y = 200\left(\frac{1}{2}\right)^{\frac{1}{432}x}$

$$\boxed{A = 200\left(\frac{1}{2}\right)^{\frac{t}{432}}}$$

Radioactive decay.

$$A = A_0\left(\frac{1}{2}\right)^{t/t_{1/2}}$$

A_0 = initial sample size $\quad t_{1/2}$ = the half-life.

$t_{1/2}$ = time it takes for ½ of a radioactive sample to decay into other particles.

$t_{1/2}$ = time it takes for 1/2 of a radioactive sample to decay into other particles.

$y = a(3)^{bx}$ $\frac{1}{b} = 6$ $b = \frac{1}{6}$

$y = 3^x$

x	y
0	1
1	3
2	9

x	y
0	4000
6	12000
12	36000

x = # of years after 2006

∴ 2006 means x = 0

x = 6, year = 2006 + 6 = 2012

x = 12, year = 2006 + 12 = 2018

Create the Equation

1) The population of a town triples every 6 years. Suppose that 4000 people lived in the town in the year 2006.

a) equation

$y = 4000(3)^{\frac{1}{6}x}$

HE by 6

VE by 4000

$P = P_0(3)^{t/6}$

P = population

P_0 = initial population

t = # of years after 2006.

b) How many people would be living in the town in 2050?

t = years after 2006 = 2050 − 2006 = 44

$P = 4000(3)^{44/6} = 12\,616\,799$ people.

2) A bacterial culture doubles every 2 hours. This culture had 22 000 bacteria at time $t = 0$.

a) equation

$y = 2^x$

$t_{1/2}$ ← doubling time

time since "zero time"

$P = P_0(2)$

initial population

$P = 22000(2)^{t/2}$

b) How many bacteria would be in the culture after 5 hours?

$P = 22000(2)^{5/2} = 124\,450.7935 = 124\,451$

3) The half-life of a radioactive sample is 4 hours. The sample size was originally 60 g.

a) equation

$t_{1/2}$ A_0

$A = A_0\left(\frac{1}{2}\right)^{t/t_{1/2}}$ ∴ $A = 60\left(\frac{1}{2}\right)^{t/4}$

b) How many grams would be in the sample after 11 hours?

t = 11 h

$A = 60\left(\frac{1}{2}\right)^{11/4} = 8.91905..$

$= 8.9$ g.

x	y
0	1

depth in m

d	% of light
0	100
1	95 0.95×100
2	95% × 95 = 0.95×95

0.95 × 0.95 × 100

4) For every metre that you descend into water, 5% of light is blocked.

(If you start with 100% of the light and 5% is blocked, what percentage of light do you still have?)

a) equation

$A = 100(0.95)^d$

95%

b) What percentage of light would still pass through the water, at a depth of 15 metres?

$A = 100(0.95)^{15} = 46.3\%$

5) $5000 is invested at 7.2% compounded **annually**.

a) equation

$c = 1 + \frac{7.2}{100}$

$= 1.072$

t	A
0	5000
1	(1.072)(5000)
2	(1.072)(1.072)(5000)

b) How much money would you have after 3 years?

$A = 5000(1.072)^3$

$= \$6158.63$

$A = A_0(1.072)^{t/t_{1/2}}$

$A = 5000(1.072)^t$

7.2% compounded quarterly 4x per year

n = # of compounds / year

$A = A_0(1 + \ldots)^{nt}$

7.2% compounded quarterly

$A = A_0\left(1+\frac{r}{n}\right)^{nt}$

0.072

$A = A_0\left(1+\frac{r}{4}\right)^{4t}$

n = # of compounds / year

r = annual rate expressed as a decimal.

7.3 Solving Exponential Equations

Remember the rules for working with exponents:

$a^m a^n = a^{m+n}$ $a^2 \times a^7 = a^{2+7} = a^9$

$\left(a^m\right)^n = a^{m\times n}$ $(a^2)^7 = a^{2\times 7} = a^{14}$

$(ab)^m = a^m b^m$ NOT $a \times b^m$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$\frac{a^m}{a^n} = a^{m-n}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $a^{2/3}$

$m \neq 1$
$n \in$ Naturals,
1, 2, 3, 4..

$a^{\frac{1}{2}} = \sqrt[2]{a^1}$
$a^{1/3} = \sqrt[3]{a^1}$

Exponential equations are ones where the ***variable is in the exponent***. We can solve these equations by

- Writing the left side of the equation and the right side of the equation so they each use the same base.
- Then, we use the fact that if $a^x = a^y$, it forces $x = y$, to finish solving the equation.

$4^{2x} = 8^{3x-1}$ $4 = 2^2$ $8 = 2^3$

$(2^2)^{2x} = (2^3)^{3x-1}$

$2^{4x} = 2^{9x-3}$

$4x = 9x - 3$
$-9x \quad -9x$

$\frac{-5x}{-5} = \frac{-3}{-5}$

$\boxed{x = 3/5}$

Example

$$8^{4x-1} = \left(\frac{1}{2}\right)^{x+5}$$

To Try

1. Rewrite the expressions so they have the same base, then solve the equation.

a) $\left(\frac{1}{25}\right)^{4x} = (125)^{3x+2}$

b) $(8)^{2x+7} = 16^{4x+2}$

c) $16^{3x} = 8^{3x-1}64^{x}$

2.For how long does one need to invest $2000 in an account that earns 6.1% compounded quarterly, before it increases in value to $2500?

$$A = P(1+i)^n$$

P = principal amount deposited
i =interest rate per compounding period, in decimal form
n = number of compounding periods

3. The population of a town triples every 6 years. If 4000 people lived there in 2009, how many will be in the town 2030? (Round down to the nearest whole person.)