Chapter 6 Note Package

February 28, 2022 8:02 AM

Chapter 6: Trigonometric Identities

6.1 Trigonometric Identities

$2x+4=2(x+2)$
let $x=3$
$2(3)+4=2(3+2)$
$10=10$ ✓

In this chapter we talk about trigonometric ***identities***. Trigonometric identities look like trigonometric equations, but there's a difference.

identities	equations
$2x+4=2(x+2)$	$2x+3=9$
$2x+4=2x+4$ ✓	$x=3$ only value of x that satisfies this equation.
$\frac{(x^2-9)}{x+3}=x-3$	$x^2+5x=-6$
$\frac{(x+3)(x-3)}{x+3}=x-3$	$x^2+5x+6=0$
$x-3=x-3$	$(x+2)(x+3)=0$
	$x+2=0$ $x+3=0$
	$x=-2$ $x=-3$
$\csc x=\frac{1}{\sin x}$ verify	$\sin x=0.53$
	$x=\sin^{-1}(0.53)$
	$=32.0°$
	$x=180-32$
	$=148°$

$2x+3=9$
$-3\quad -3$
$2x=6$
$x=\frac{6}{2}=3$

You can't $\div 0$, so $x=-3$ is a "non-permissible value"

x^2-3^2

$\sin x \neq 0$
$x\neq 0, 180°, 360° \ldots$
$x\neq n(180°);\ n\in I$

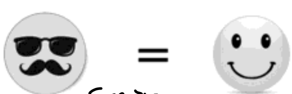
$x=32.0°+360°n$
$=148°+360°n$
$n\in I$

Identity - an equation that is true for **ALL** permissible values

When we are given an identity to prove, we see different expressions on the left-side and right-side of the equation. Proving the identity means we must ***change*** the expressions so that we end up with the SAME expression on both sides of the equation.

Our tools to do this are:

- algebra skills (getting common denominator, combining terms, factoring)
- basic identity substitutions



$y=\tan x$ $\quad y=\frac{\sin x}{\cos x}$

$\tan x=\frac{\sin x}{\cos x}$ not valid for $\cos x=0$

$x\neq 90°, 270°, 450°$

$\neq 90°+180°n,$
$n\in I$

$\tan 30°=\frac{\sin 30°}{\cos 30°}$

$0.57735=0.57735$

verified

We will be verifying and proving trigonometric identities.

- ***Verifying*** an identity means we show it *seems* true. Done by:
  - *substituting* in a specific value and confirming that, for that value,

We will be verifying and proving trigonometric identities.

- ***Verifying*** an identity means we show it *seems* true. Done by:
    - *substituting* in a specific value and confirming that, for that value, the left and right sides of the identity are equal
    - *graphing* the left and right sides of the identity separately, and confirming that the graphs are exactly the same in that window
- ***Proving*** an identity means using algebra and/or Basic Identities to change the form of one or both sides of the identity, until the two sides are exactly the same.

Example

Verify the given identity, for the value $x = \frac{\pi}{5}$ $\qquad \sec x = \frac{\tan x}{\sin x}$

$\sec\frac{\pi}{5}$ $\qquad \sec\left(\frac{\pi}{5}\right) = \frac{\tan \pi/5}{\sin \pi/5}$ $\qquad (\sin x \neq 0)$

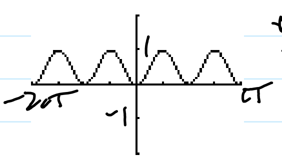
$\frac{1}{\cos\left(\frac{\pi}{5}\right)}$ $\qquad \frac{1}{\cos\frac{\pi}{5}} = \frac{\tan \pi/5}{\sin \pi/5}$

$1.236 = 1.236$

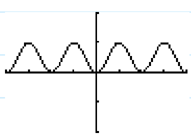
Example

Verify graphically: $\qquad \frac{\sin^3 x}{\sin x} = \sin^2 x$

$y = \frac{\sin^3 x}{\sin x}$ $\qquad y = \sin^2 x$

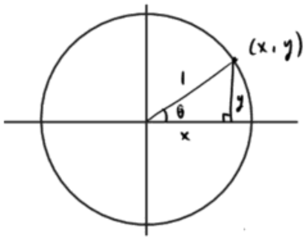


$y = \frac{\sin^3 x}{\sin x}$



$y = \sin^2 x$

Basic Identities



Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

Quotient Identities $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Addition Identities

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double Angle Identities

$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) = 1 - 2\sin^2\theta$

Practice Using Basic Identities

Match the expressions on the left with those on the right-hand column. Put the letter of the expression that matches in the blank provided. Each gets used exactly once.

G	1.	$\frac{\sin B}{\cos B} = \tan B$	A.	1
H	2.	$\frac{1}{\cos B} = \sec B$	B.	$\sin^2 B$
C	3.	$\csc^2 B = 1 + \cot^2 B$	C.	$\cot^2 B + 1$ ($1 + \cot^2 B$)
A	4.	$\sin^2 B + \cos^2 B = 1$	D.	$\cos^2 B$
J	5.	$\cot B \sin B \left(\frac{\cos B}{\cancel{\sin B}}\right)\left(\frac{1}{\cancel{\sin B}}\right) = \cos B$	E.	$1 + \tan^2 B$
F	6.	$\frac{\cos B}{\sin B} = \cot B$	F.	$\cot B$
B	7.	$1 - \cos^2 B$	G.	$\tan B$
K	8.	$\frac{\cos^2 B}{1+\sin B} = \frac{1-\sin^2 B}{1+\sin B} = \frac{\cancel{(1+\sin B)}(1-\sin B)}{\cancel{1+\sin B}}$	H.	$\sec B$
E	9.	$\sec^2 B$	I.	$\csc B$
I	10.	$\frac{1}{\sin B}$	J.	$\cos B$
L	11.	$\frac{\cos B}{\cot B}$ $(\cos B) \times \left(\frac{1}{\cot B}\right)$	K.	$1 - \sin B$
D	12.	$1 - \sin^2 B$	L.	$\sin B$

$1^2 - \sin^2 B$

$a^2 - b^2$

$(\cos B)(\tan B)$

$(\cancel{\cos B})\left(\frac{\sin B}{\cancel{\cos B}}\right) = \sin B$

$\sec\theta = \frac{1}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

$\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta}$

$\frac{\cos\theta}{\sin\theta} = \cot\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\rightarrow 1 + \cot^2\theta = \csc^2\theta$

$\tan^2\theta + 1 = \sec^2\theta.$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$\cot B = \frac{\cos B}{\sin B}$

$\frac{1}{\cot B} = \frac{\sin B}{\cos B}$

$(= \tan B).$

More Practice

Simplify each expression below. Look for substitutions you can make, using basic identities. Your final answer should contain no more than one trigonometric function.

1. $\frac{\cos^2\theta}{\sin^2\theta}$ — $\cot^2\theta$

2. $\tan\theta\sec\theta\cos\theta$ — $\tan\theta$

3. $1-\cos^2\theta$ — $\sin^2\theta$

4. $\cos^2\theta-1$ — $-\sin^2\theta$

5. $1+\tan^2\theta$ — $\sec^2\theta$

6. $\sin^2\theta+\cos^2\theta+1$ — 2

Pyth. IDs
$\sin^2\theta+\cos^2\theta=1$
$\tan^2\theta+1=\sec^2\theta$
$\tan^2\theta=\sec^2\theta-1$

discuss

7. $\csc^2\theta-\cot^2\theta$ — 1

$(1+\cot^2\theta)-\cot^2\theta$
$=1$

8. $(\sin^2\theta+\cos^2\theta+\tan^2\theta)$ — $\sec^2\theta$

$1+\tan^2\theta$

9. $\frac{\sin^2\theta+\sin\theta}{\cos\theta+\cos\theta\sin\theta}$ — $\tan\theta$

$\frac{\sin\theta(\cancel{\sin\theta+1})}{\cos\theta(\cancel{1+\sin\theta})}=\frac{\sin\theta}{\cos\theta}$
$=\tan\theta$

10. $\frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\tan^2 x}}$ — $\frac{\cos x}{\sec x}$ — $\cos^2 x$

$\sqrt{\cos^2 x}=\sqrt{1-\sin^2 x}$
$\sqrt{1+\tan^2 x}$
$\sqrt{\sec^2 x}$
$=\sec x$

$\frac{\cos x}{\frac{1}{\cos x}}=\cos x\times\frac{\cos x}{1}=$

11. $1-\sec^2 x$ — $\boxed{-\tan^2 x}$

$1-\sec^2 x$
$-1(-1+\sec^2 x)$
$-1(\sec^2 x-1)$
$-1(\tan^2 x)$

12. $\sec^2 x-1$ — $\tan^2 x$

6.0 Algebra Skills Used in Chapter 6

Multiplying Trigonometric Expressions

1. $\sin x(2\sin x-1)$ $2\sin^2 x-\sin x$

2. $(\cos x+2)(\cos x-7)$ $\cos^2 x-7\cos x+2\cos x-14$

$\cos^2 x-5\cos x-14$

3. $(\cos x-3)^2$ $(\cos x-3)(\cos x-3)$

$\cancel{\cos^2 x-3^2}$ $\cos^2 x-6\cos x+9$

NOT $\cos^2 x-9$

Factoring Trigonometric Expressions

Greatest Common Factor

1. $\sin^2 x-3\sin x$

$\frac{\sin^2 x}{\sin x}$ $\frac{-3\sin x}{\sin x}$

$\sin x(\sin x-3)$

2. $5\tan^2 x+15\tan x$ $\frac{5\tan^2 x}{5\tan x}$ $\frac{15\tan x}{5\tan x}$ $5\tan x(\tan x+3)$.

Difference of Perfect Squares $a^2-b^2=(a+b)(a-b)$

1. $\sin^2 x-1$

$(\sin x)^2-(1)^2$

$(\sin x+1)(\sin x-1)$.

2. $1-\tan^2 x$

$(1+\tan x)(1-\tan x)$

$2x - 1 = -2$

Trinomials

1. $2\cos^2 x + \cos x - 1$

$\boxed{2} \times \boxed{-1} = -2$

$\boxed{2} + \boxed{-1} = +1$

$(2\cos x - 1)(1\cos x + 1)$

$+1\cos x$

$2\cos^2 x + 2\cos x - 1\cos x - 1$

$2\cos x(\cos x + 1) - 1(\cos x + 1)$

$(2\cos x - 1)(\cos x + 1)$.

2. $3\sin^2 x + 2\sin x - 1$

$(3\sin x - 1)(\sin x + 1)$

$3\sin^2 x + 3\sin x - 1\sin x - 1$

$3\sin x(\sin x + 1) - 1(\sin x + 1)$

$3x - 1 = -3$

$\boxed{+3} \times \boxed{-1} = -3$

$\boxed{+3} + \boxed{-1} = +2$

Adding/Subtracting Trigonometric Terms

$1\sin 45^\circ + 6\sin 45^\circ$

$7\sin 45^\circ$

We can only add ***like terms***

- Terms must contain the same angle
- Terms must use the same trigonometric function

Which of these terms can be combined?

$4\sin x + 3\cos x$ can't add

$2\sin x + 5\sin 2x$ can't add.

$3\sin x + 4\sin x$

$7\sin x$

$2\sin^2 x + 4\sin x$ can't add

$= 2\sin x(\sin x + 2)$

Errors to Avoid

$2a^2 + 4a$

~~can't~~ combine

Omitting the angle

$\cos + 4\cos$ These terms contain no angle - they don't mean anything!

$\cos x + 4\cos x = 5\cos x$

Incorrect cancelling

You can never "cancel" the angle, or any part of the angle, in a trigonometric expression.

$\dfrac{\cos x}{x}$

cos

$\dfrac{\tan 2x}{2}$

$\dfrac{2\tan x}{2}$

$= \tan x$.

More incorrect cancelling
You can NEVER cancel just a portion of a factor.

$\dfrac{\cos x+1}{\cos x}=$ You **CANNOT** cancel the "$\cos x$" on the top with the "$\cos x$" on the bottom!

top

$\dfrac{(\cos x+1)(\cos x-1)}{(\cos x+1)}=$ You **CAN** cancel the "$\cos x+1$" factors

$\dfrac{3\;\cancel{6\sin x}\;1}{1\;\cancel{2\sin x}\cos x}=\dfrac{3}{\cos x}$

To CORRECTLY simplify a rational expression, factor it completely.
If the numerator and the denominator contain the same factor, you can reduce.

For example: $\dfrac{\sin^2 x+8\sin x+12}{\sin^2 x+\sin x-30}=\dfrac{\cancel{(\sin x+6)}(\sin x+2)}{\cancel{(\sin x+6)}(\sin x-5)}$

$\dfrac{x^2(2x)(3)}{x^2(3)}=\dfrac{2x}{1}$
$=2x$

Notice that we **cannot** reduce $\dfrac{\sin x+2}{\sin x-5}$ by canceling the $\sin x$'s, because $\sin x$ is NOT a factor of the numerator and the denominator. $\dfrac{\sin x+2}{\sin x-5}\neq\dfrac{2}{-5}$

$\dfrac{(\cancel{\sin x})(2)}{(\cancel{\sin x})(5)}=\dfrac{2}{5}$.

Distributing when you can't:

$\cos(x+y)$ This does **NOT** equal $\cos x+\cos y$!
We are not multiplying "cos" with $(x+y)$.

What this expression DOES mean is the cosine of the angle "$x+y$"

For example, consider what happens if $x=15^\circ$ and $y=28^\circ$

$\cos(15^\circ+28^\circ)=\cos(43^\circ)\doteq 0.73$ ✓

$\cos(15^\circ)+\cos(28^\circ)\doteq 1.85$ ✗

This shows us that $\cos(x+y)\neq\cos x+\cos y$

6.2 Sum, Difference, and Double-Angle Identities

Sum/Difference Identities

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\qquad$ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\qquad$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$A-B$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double Angle Identities

$\sin(2\theta)=2\sin\theta\cos\theta$ $\qquad$ $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $\qquad$ $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$\cos(2\theta)=2\cos^2\theta-1$

$\cos(2\theta)=1-2\sin^2\theta$

$195°$ $\quad$ $45°$

$195° = 150° + 45°$

$195° = 135° + 60°$

Examples

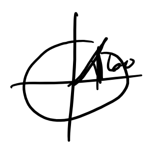
Find the exact value of the following expressions.

a) $\cos 195° =$

b) $\sin\left(\dfrac{5\pi}{12}\right)=$

c) $\sin 50°\cos 10° + \cos 50°\sin 10° =$ $\quad$ $\sin(50°+10°) = \sin 60°$

$= \boxed{\dfrac{1}{2}}$

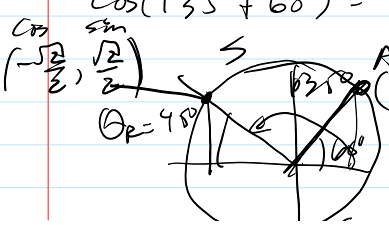


a) $\cos 195°$) This can be expressed as:

$\cos(135°+60°)$ $\qquad$ $\cos(240°-45°)$

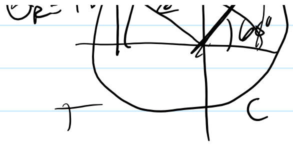
$\cos(150°+45°)$

$\cos(135°+60°) = \cos 135°\cos 60° - \sin 135°\sin 60°$



$\cos 135° = -\dfrac{\sqrt{2}}{2}$ $\qquad$ $\sin 135° = \dfrac{\sqrt{2}}{2}$

$\cos 60° = \dfrac{1}{2}$ $\qquad$ $\sin 60° = \dfrac{\sqrt{3}}{2}$



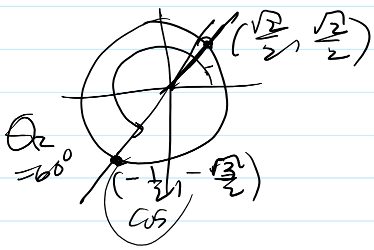
$\cos 60^\circ = \frac{1}{2}$ $\quad \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos(195^\circ) = \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{-\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

$= \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$

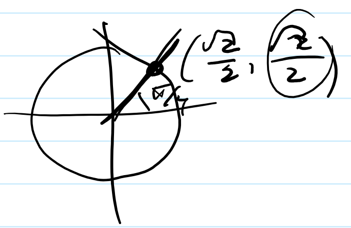
$\cos(240^\circ - 45^\circ) = \cos 240^\circ \cos 45^\circ + \sin 240^\circ \sin 45^\circ$

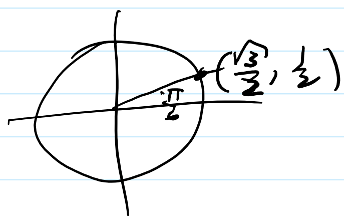


$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{-\sqrt{2}}{4} + \frac{-\sqrt{6}}{4} = \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$

b) $\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$





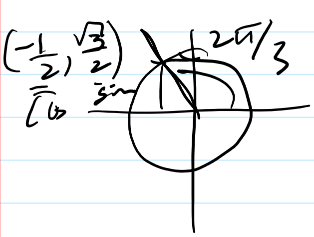
$\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4} \cdot \cos\frac{\pi}{6} + \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{6}$

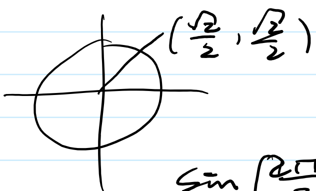
$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ $\quad = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

odd.

$\frac{2\pi}{12}, \frac{3\pi}{12}, \frac{4\pi}{12}$ $\quad = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}} = \sin\frac{5\pi}{12}$

odd $\to \frac{5\pi}{12} = \left(\frac{8\pi}{12} - \frac{3\pi}{12}\right) = \left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$





$\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \cos\frac{2\pi}{3}\sin\frac{\pi}{4}$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{4} - -\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \frac{\sqrt{2}}{4} - \frac{?}{2} = \frac{\sqrt{6}}{4} + \frac{?}{4} = \boxed{\frac{?}{4}}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

Write each expression in a simpler form, using identities. Give exact value, if possible.

a) $2\sin 15^\circ \cos 15^\circ =$ $\sin[2(15^\circ)]$
$= \sin 30^\circ$
$= \frac{1}{2}$

b) $\cos^2(\pi/8) - \sin^2(\pi/8) =$ $\cos(2 \times \frac{\pi}{8})$
$= \cos(\frac{\pi}{4})$
$= \frac{\sqrt{2}}{2}$

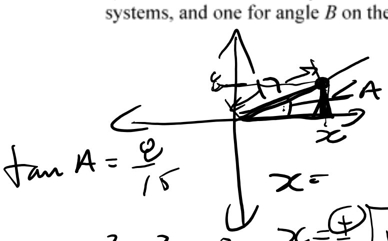
c) $1 - 2\sin^2(8) =$ 8 radians
$\cos(2\times 8) = \boxed{\cos 16}$

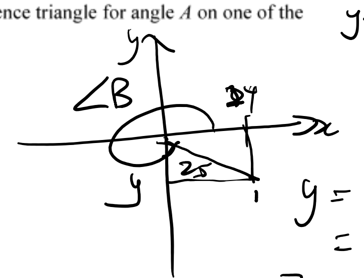
d) $10\cos^2(x) - 5 =$ $5(2\cos^2 x - 1)$
$= 5(\cos 2x) = \boxed{5\cos 2x}$

cos A   sin B = ?

Given that $\sin A = \frac{8}{17}$, where A is a Q1 angle; and $\cos B = \frac{24}{25}$, where B is a Q4 angle. $x > 0$, $y < 0$

a) Draw two coordinate systems. Sketch a reference triangle for angle A on one of the systems, and one for angle B on the other.





$\tan A = \frac{8}{15}$

$x^2 + y^2 = r^2$ $x = \pm\sqrt{r^2 - y^2}$
$x = +\sqrt{17^2 - 8^2} = 15$

$y = -\sqrt{r^2 - x^2}$
$= -\sqrt{25^2 - 24^2} = -7$

$\sin B = \frac{-7}{25}$

$\tan B = \frac{-7}{24}$

b) Use identities to find the exact value of:

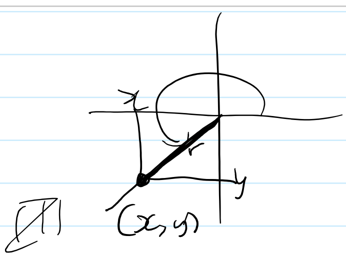
$\sin(A - B)$ $\cos A = \frac{15}{17}$

$\sin(A - B)$
$= \sin A\cos B - \cos A \sin B$
$= \left(\frac{8}{17}\right)\left(\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(\frac{-7}{25}\right)$
$= \boxed{\frac{297}{425}}$

$\tan(2A)$
$= \frac{2\tan A}{1 - \tan^2 A} = \frac{2(8/15)}{1 - (8/15)^2} = \frac{16/15}{1 - 64/225}$
$= \frac{\frac{16}{15}}{\frac{225}{225} - \frac{64}{225}} = \frac{16/15}{161/225}$
$= \frac{16}{15} \times \frac{225}{161} = \boxed{\frac{240}{161}}$

$\sin 2B$
$= 2\sin B\cos B$
$= 2\left(-\frac{7}{25}\right)\left(\frac{24}{25}\right)$
simplify

$\cos 2A$
$= \cos^2 A - \sin^2 A$
$= \left(\frac{15}{17}\right)^2 - \left(\frac{8}{17}\right)^2$
$= \frac{225 - 64}{289} =$ simplify.



$r^2 = x^2 + y^2$, $y^2 = \pm\sqrt{r^2 - x^2}$
$x^2 = \pm\sqrt{r^2 - y^2}$

6.3 Proving Identities

When we prove identities:

- Step by step, use algebra and/or Basic Identities to change the way either the left- hand side (LHS) or the right-hand side (RHS) looks. Change the more complicated side.
- Think of the " = " sign separating the LHS and RHS as a barrier. Don't take terms from one side of the equals sign to the other.
- When the LHS and the RHS look exactly the same, the identity is proven.

Strategies for Proofs

- Write each step directly below the previous one.
- Don't skip steps – aim to be as CLEAR as possible.
- See if there's any factoring you can do, especially GCF or difference of squares.
- Don't cancel anything, unless you have identical factors on the top and bottom of an expression.
- If rational expressions are added /subtracted together, get a common denominator so you can combine the expressions and simplify.
- If possible, substitute known identities to simplify expressions.
- If the LHS and RHS look as below, where they are ***almost*** reciprocals of each other, multiply one side, top and bottom, by the conjugate of the binomial. Then use a Pythagorean identity to simplify further.

$$\frac{\cos y}{1-\sin y} = \frac{1+\sin y}{\cos y}$$

$$\frac{\cos y}{(1-\sin y)} \times \frac{(1+\sin y)}{(1+\sin y)}$$

$(1-\sin y)(1+\sin y)$

$1-\sin^2 y$

$\sin^2 y+\cos^2 y=1$

$1-\sin^2 y=\cos^2 y$

Try these strategies to prove the identities on the following pages.

$$\frac{\cos y(1+\sin y)}{1-\sin^2 y}$$

$$\frac{\cos y(1+\sin y)}{\cos^2 y}$$

$$\frac{1+\sin y}{\cos y} = \frac{1+\sin y}{\cos y}$$

$-(\sin y-1)$

$1-\sin y$

$-1(-1+\sin y)$

$-1(\sin y-1)$

1. $(\sin x+\cos x)^2 \quad = \quad 1+2\sin x\cos x$

$\sin^2 x+\cos^2 x = 1 \qquad -\cos^2 x$
$\sin^2 x - 1 = -\cos^2 x$

2. $\tan^2 x\sin^2 x-\tan^2 x \quad = \quad -\sin^2 x$

$\tan^2 x(\sin^2 x-1)$
$\tan^2 x(-\cos^2 x)$
$\frac{\sin^2 x}{\cancel{\cos^2 x}}(-\cancel{\cos^2 x})$
$-\sin^2 x$

$= -\sin^2 x$

x^4+2x^2+1
$(x^2+1)(x^2+1)$

3. $\sec^4 x \quad = \quad \tan^4 x+2\tan^2 x+1$

$(\tan^2 x+1)(\tan^2 x+1)$
$(\tan^2 x+1)^2$
$(\sec^2 x)^2$
$\sec^4 x$

4. $(3-3\sin x)(3+3\sin x) \quad = \quad 9\cos^2 x$

$9-9\sin^2 x$
$9(1-\sin^2 x)$
$9\cos^2 x$

becomes
$1-\sin^2 x=\cos^2 x$

$= \quad 9\cos^2 x$

FOIL

1. $(\sin x+\cos x)^2 = 1+2\sin x\cos x$

$(\sin x+\cos x)(\sin x+\cos x)$

$\sin^2 x+\sin x\cos x+\sin x\cos x+\cos^2 x$

$\sin^2 x+2\sin x\cos x+\cos^2 x$

$\sin^2 x+\cos^2 x+2\sin x\cos x$

$1+2\sin x\cos x = 1+2\sin x\cos x$

$\frac{3}{3} + \frac{5}{7}$

Combine these

5. $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} \quad = \quad 2\csc^2 x$

$\csc x = \frac{1}{\sin x}$

$\frac{1 \times (1-\cos x)}{(1+\cos x)(1-\cos x)} + \frac{1 \times (1+\cos x)}{(1-\cos x)(1+\cos x)}$

$\frac{1-\cos x}{1-\cos^2 x} + \frac{1+\cos x}{1-\cos^2 x}$

$\frac{2}{1-\cos^2 x} \to \frac{2}{\sin^2 x}$

$2\left(\frac{1}{\sin^2 x}\right)$

$\frac{2}{\sin^2 x}$

$= \frac{2}{\sin^2 x}$

6. $\frac{\cos y}{1-\sin y} \quad = \quad \frac{1+\sin y}{\cos y}$

7. $2\sec x \quad = \quad \frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x}$

$\sec x = \frac{1}{\cos x}$

$\frac{2}{\cos x}$ ✓

LCD $(1+\sin x)(\cos x)$

$\frac{\cos x}{1+\sin x} \times \frac{\cos x}{\cos x} + \frac{(1+\sin x)}{\cos x} \times \frac{(1+\sin x)}{(1+\sin x)}$

$\frac{\cos^2 x + (1 + 2\sin x + \sin^2 x)}{(1+\sin x)(\cos x)}$

$\frac{(1-\cancel{\sin^2 x}) + (1 + 2\sin x + \cancel{\sin^2 x})}{(1+\sin x)(\cos x)}$

$\frac{2+2\sin x}{(1+\sin x)\cos x}$

$\frac{2(1+\cancel{\sin x})}{(\cancel{1+\sin x})\cos x}$

$\frac{2}{\cos x}$ ✓

8. $\frac{\sin x}{1+\cos x} \quad = \quad \frac{1-\cos x}{\sin x}$

6.4 Solving Trigonometric Equations Using Identities

Some trigonometric equations cannot be solved until they are re-written in a different form, using trigonometric identities.

Example

Algebraically solve this equation, giving the ***general solution***, in radian measure.

$$2\sin x = 9 - 4\csc x$$

$\csc x = \frac{1}{\sin x}$ LCD: $(1)(\sin x)$

$2x^2 - 9x + 4$ $2 \times 4 = 8$
$2x^2 - 8x - 1x + 4$
$2x(x-4) - 1(x-4)$
$(2x-1)(x-4)$

$\frac{\sin x}{\sin x} \cdot \frac{2\sin x}{1} = \frac{\sin x \, 9}{\sin x \, 1} - \frac{4}{\sin x}$

$\frac{2\sin^2 x}{\sin x} = \frac{9\sin x}{\sin x} - \frac{4}{\sin x}$

$(\sin x)\left(\frac{2\sin^2 x}{\sin x}\right) = \left(\frac{9\sin x - 4}{\sin x}\right)\sin x$

FACTOR

$2\sin^2 x = 9\sin x - 4$
$2\sin^2 x - 9\sin x + 4 = 0$
$(2\sin x - 1)(\sin x - 4) = 0$
$2\sin x - 1 = 0$ $\sin x - 4 = 0$
$\sin x = 4$ no solution
$2\sin x = 1$
$\sin x = \frac{1}{2}$

$\sin x = +\frac{1}{2}$
$x = \frac{\pi}{6}, \frac{5\pi}{6}$
period $= 2\pi$
$x = \frac{\pi}{6} + 2\pi n, n \in I$
$x = \frac{5\pi}{6} + 2\pi n, n \in I.$

Example

Algebraically solve this equation

$$\cos 2x + \cos x = -1 \text{, for } 0° \le x < 360°$$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$

GCF Factor

$(2\cos^2 x - 1) + \cos x = -1$
$2\cos^2 x - 1 + \cos x + 1 = 0$
$2\cos^2 x + \cos x = 0$
$\cos x(2\cos x + 1) = 0$
$\cos x = 0$ $2\cos x + 1 = 0$
$\cos x = -\frac{1}{2}$

$\cos x = 0$
$x = 90°, 270°$
$\cos x = -\frac{1}{2}$
$x = 120°$
$= 240°$

$\cos 60° = \frac{1}{2}$

$x = 90°, 120°, 240°, 270°$

$\cos^2 x + 3\cos x - 1\cos x - 3 = 0$
$\cos x(\cos x + 3) - 1(\cos x + 3)$ $\times -3 = -3$
$\boxed{-1} \times \boxed{3} = -3$

$1\cos^2 x + 2\cos x - 3 = 0$ $\boxed{-1} + \boxed{3} = +2$
$(\cos x \,\_\_)(\cos x \,\_\_) = 0$

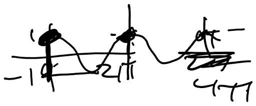
Try

Algebraically solve this equation

$$2\cos x + 1 - \sin^2 x = 3 \text{, for } 0 \le x < 2\pi$$

$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

$2\cos x + 1 - (1 - \cos^2 x) = 3$
$2\cos x + 1 - 1 + \cos^2 x = 3$
$\cos^2 x + 2\cos x - 3 = 0$
$(\cos x - 1)(\cos x + 3) = 0$



$\cos^2 x + 2\cos x - 3 = 0$ $\qquad$ $-\pi \quad \cancel{\ } \ 2\pi \quad \cancel{\ }$ 4π

$(\cos x - 1)(\cos x + 3) = 0$

$\cos x - 1 = 0$ $\qquad$ $\cos x + 3 = 0$

$\cos x = 1$ $\qquad$ $\cos x = -3$ ← NO solution.

$x = 0, 2\pi, 4\pi$

[general solution: $x = 2\pi n$, $n \in I$].