Chapter 4 Note Package

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Chapter 4: Trigonometry and the Unit Circle

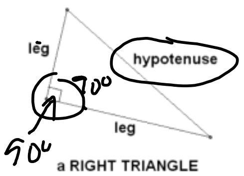
4.0 Trigonometry Review

Trigonometry is the study of triangles and trigonometric functions. First, we review some trigonometry dealing with triangles.

Triangles

- have three angles, the measures add up to 180°
- longest side of triangle is across from the largest angle
- shortest side of a triangle is across from the smallest angle

- ***right triangles*** are triangles that have a ***right angle*** (90°)
  - ***hypotenuse*** is the longest side of a right triangle
  - other sides of the triangle are often called ***legs***
  - hypotenuse is always **across** from the right angle
  - in a right triangle, we can use the ***Pythagorean Theorem***



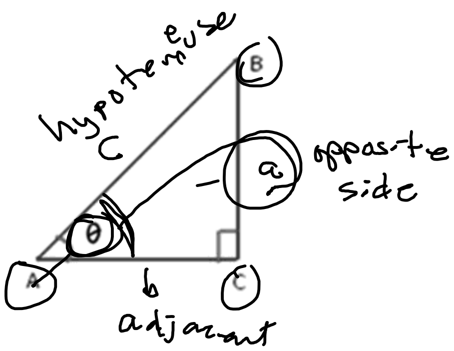
If the two legs of a right triangle are called a and b, and the hypotenuse is called c, then $a^2 + b^2 = c^2$.

$p^2 = r^2 + q^2$

hyp.

$a^2 = b^2 + c^2$

Look at the right triangle shown below. The angle by point A is labeled with the Greek letter θ, read "theta." Angles are very commonly labeled with the letter θ.



Which side is the hypotenuse? c

Which side is opposite θ? a

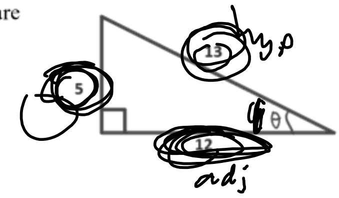
Which side is adjacent to θ? b

next to

Sine (sin)
Cosine (cos)
Tangent (tan)

When we know the lengths of the sides of a right triangle, we can calculate **ratios** that compare the lengths of two different sides.

First, we label the hypotenuse, and the sides that are opposite and adjacent to angle θ .



There are six different ratios one can create. We'll leave the ratios in fractional form.

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{5}{13} \quad \cos\theta = \frac{adjacent}{hypotenuse} = \frac{12}{13} \quad \tan\theta = \frac{opposite}{adjacent} = \frac{5}{12}$$

cosecant θ secant cotangent.

$$\csc\theta = \frac{hypotenuse}{opposite} = \frac{13}{5} \quad \sec\theta = \frac{hypotenuse}{adjacent} = \frac{13}{12} \quad \cot\theta = \frac{adjacent}{opposite} = \frac{12}{5}$$

Cosecant is the reciprocal of sine.

These ratios are called the ***trigonometric ratios***. Knowing them makes it possible to find the measure of each angle in the triangle.

Primary Trigonometric Ratios

$$SINE = \sin\theta = \frac{opposite}{hypotenuse} = \frac{O}{H}$$

$$COSINE = \cos\theta = \frac{adjacent}{hypotenuse} = \frac{A}{H}$$

$$TANGENT = \tan\theta = \frac{opposite}{adjacent} = \frac{O}{A}$$

Reciprocal Trigonometric Ratios

$$COSECANT = \csc\theta = \frac{hypotenuse}{opposite} = \frac{H}{O} = \frac{1}{\sin\theta}$$

$$SECANT = \sec\theta = \frac{hypotenuse}{adjacent} = \frac{H}{A} = \frac{1}{\cos\theta}$$

$$COTANGENT = \cot\theta = \frac{adjacent}{opposite} = \frac{A}{O} = \frac{1}{\tan\theta}$$

C S
90° 180°

Supplementary angles add to 180°

$A+B+C=180°$

$28°+90°+C=180°$

$C=62°$ b and angle

Two angles that add to 90° are called complementary angles.

Example Find the measure of each side and angle (correct to nearest degree) in the right triangle shown below.

$\tan C = \frac{c}{a} = \frac{15}{8}$

$C = \tan^{-1}(\frac{15}{8})$

$= 62°$

$b^2 = a^2 + c^2$

$b^2 = 8^2 + 15^2$

$b^2 = 289$

NOT $c^2 = a^2 + b^2$

To get angle A, use a trig ratio

$\tan A = \frac{a}{c} = \frac{8}{15} (= 0.5833)$

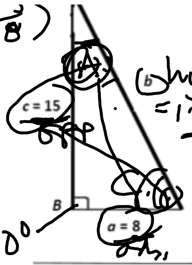
$C = \tan^{-1}\left(\frac{15}{8}\right)$

$= 62^\circ$

Notice

$\sin C = \frac{15}{17}$

$\cos C = \frac{8}{17}$ 90°



hyp = 17

$b^2 = 8^2 + 15^2$

$b^2 = 289$

$b = \sqrt{289}$

$b = 17$

trig ratio

$\tan A = \frac{a}{c} = \frac{8}{15}$ $(= 0.5333)$

$A = \tan^{-1}\left(\frac{8}{15}\right) = 28.07248...$

$A = 28^\circ$

$\sin A = \frac{8}{17}$

$A = \sin^{-1}\left(\frac{8}{17}\right) = 28.07..$

$= 28^\circ.$

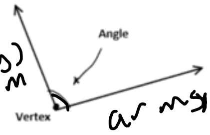
$\cos A = \frac{15}{17}$

4.1 Angles and Angle Measure

Trigonometry does a lot more than solve triangles. It can be used to analyze many ***repeating patterns*** – things like sound, light, ocean tides, and circular motion.

We start by looking carefully at **ANGLES**.

Remember, angles measure the space between two rays that meet at the ***vertex*** of the angle.

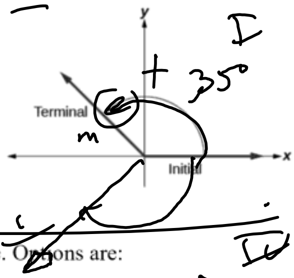


(ray) arm

arm (ray)

Angles in ***standard position***

- Have the vertex at the origin (0, 0)
- Have a specific direction of rotation, shown with an arrow.
- Have the ***initial arm*** on the positive *x*-axis
- Have the ***terminal arm*** either in one of the four quadrants, or on the *x*- or *y*-axis.



I

+ 35°

m

IV

Wherever the terminal arm is, that's how we decide what to call an angle. Options are:

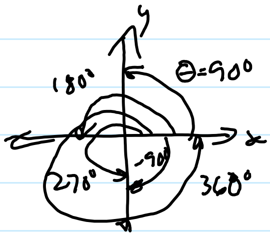
- First quadrant angle ← I angle
- Second quadrant angle
- Third quadrant angle
- Fourth quadrant angle
- Quadrantal angle

more next day!

terminal arm is along one of the axes

positive angles are formed when the rotation is counter-clockwise

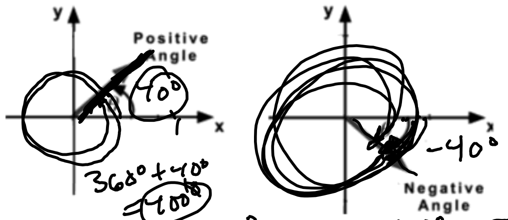
- negative angles are formed when the rotation is clockwise



anti-clockwise

positive angles start on the positive x-axis, and rotate counter-clockwise

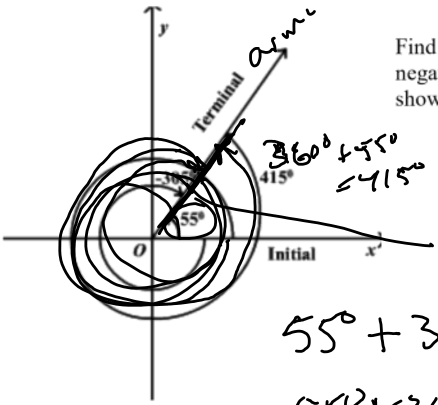
negative angles start on the positive x-axis, and rotate clockwise



$400° + 360°$
$= 760°$

$-40° + -360° = -400°$
$-400° + -360° = 760°$

Coterminal Angles are
- different in size but
- terminate in the same place



Find another positive angle and another negative angle that are coterminal to the shown angles.

an angle is co-terminal with itself.

$55° + -360° = -305°$

$55° + 360°n \quad , n \in I$

$55° + 360° \times 0$
$= 55°$

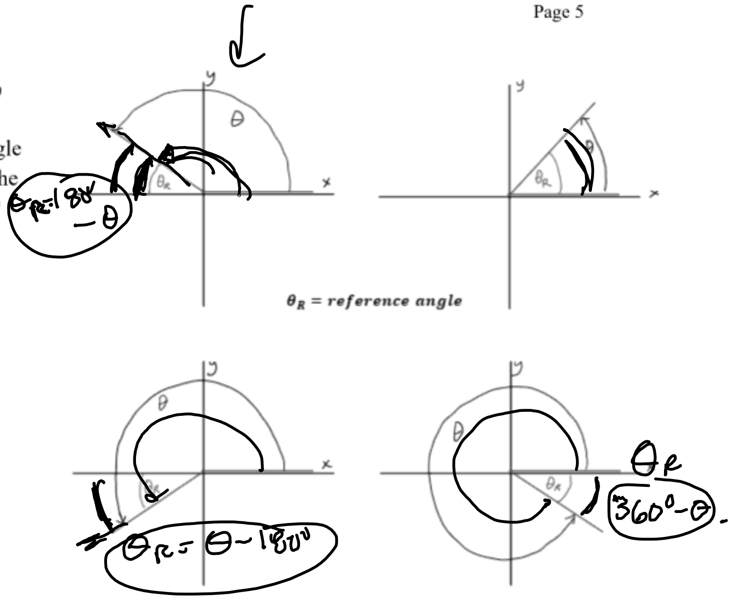
$(\ldots -3, -2, -1, 0, 1, 2, 3, \ldots)$

General form for coterminal angles – this is an expression that generates ALL the angles that are coterminal to a specific angle. Here's how we write the angles coterminal to $55°$ in general form:

$$\boxed{\theta_{ct} = \theta + 360°n \quad , n \in I}$$

reference angle θ_R of a standard position angle θ

- is the smallest angle formed between the terminal arm of θ and the x-axis
- is positive

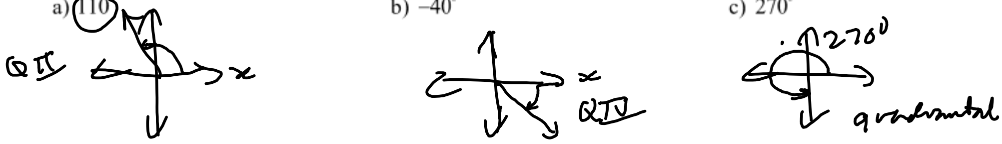


Try this

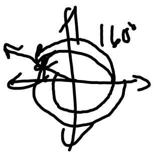
1. Draw each angle in standard position. (Estimate – you don't need to use a protractor.)

$90° \times 2$ Which quadrant??

a) $110°$ b) $-40°$ c) $270°$



2. Find a positive and a negative coterminal angle to the angle $160°$.



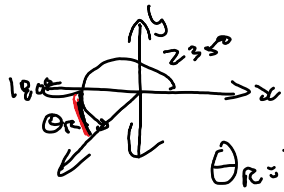
$160° + 360° = 520°$ $160° + -360° = -200°$

3. Give the general expression for ALL angles coterminal to the angle $25°$.

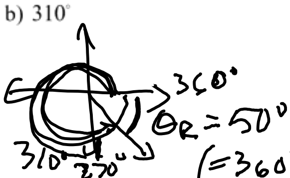
$\theta_{ct} = 25° + 360° n, \; n \in I$

4. Find the reference angle for each angle.

a) $235°$ θ_R b) $310°$ c) $110°$ θ_R



$\theta_R = 235° - 180° = 55°$

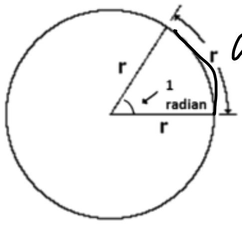


$\theta_R = 50°$ $(= 360° - 310°)$

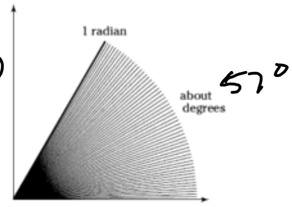
$\theta_R : 70°$. $(180° - 110°)$.

Another unit used to measure angles (besides degrees) is ***radians***. We need to know how to work with radians, as they make some calculus questions much easier (and this is PreCalculus, after all!)

1 *radian* = the measure of an angle that cuts off a length on the circle equal in length to the circle's radius.



arc length = r(radius) r

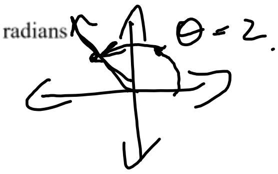


Try this

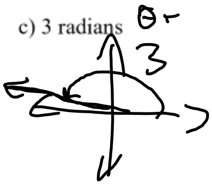
Sketch a standard position angle measuring:

a) 1 radian

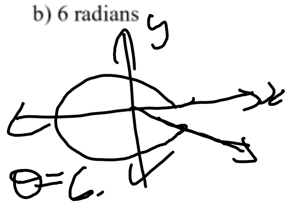
b) 2 radians



c) 3 radians



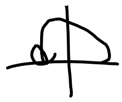
b) 6 radians



How many radians are in a full rotation? (Think about how many radius lengths will fit onto the full circumference of a circle.)

2π *radians = a complete rotation =* $360°$

π *radians = a straight angle =* $180°$.



Common Angles

Some angles are used so frequently that it is very helpful to simply KNOW their measurement in both radians and degrees.

$\frac{\pi}{2} \times \frac{180^\circ}{\pi}$

$\pi =$ 180° $2\pi =$ 360° $\frac{\pi}{2} =$ 90° $\frac{\pi}{3} =$ 60°

$\frac{180}{4}$

$\frac{\pi}{4} =$ 45° $\frac{\pi}{6} =$ 30° $0 =$ 0° $\frac{3\pi}{2} =$ 270°

Converting Units

We know that 30 minutes is the same thing as ½ an hour. But what about 7452 minutes? What is that, in hours? Here's one way to change units – multiply by a factor of "1"

Converting Angle Measure

For angles that are not the common ones listed above, we convert angle measurements between degrees and radians by multiplying by the appropriate conversion unit.

$$(\text{degrees}) \times \left(\frac{\pi}{180^\circ}\right) = \text{radians}$$

$$(\text{radians}) \times \left(\frac{180^\circ}{\pi}\right) = \text{degrees}$$

Try these

Convert from degrees to radians. Express answer correct to 2 decimal places.

$425^\circ \times \frac{\pi}{180^\circ} = 7.42$ exact: $\frac{425\pi}{180}$ $\boxed{\frac{85\pi}{36}}$

Convert from degrees to radians. ***Leave answer as a simplified fraction, in terms of π.***

-330° $\frac{-330^\circ \times \pi}{180^\circ} = -\frac{11}{6}\pi$ $-\frac{11\pi}{6}$

Convert from radians to degrees.

a) $\frac{3\pi}{8}$ $\frac{3\pi}{8} \times \frac{180^\circ}{\pi}$ $= \frac{3 \times 180^\circ}{8} = \frac{3 \times 45}{2} = 67.5^\circ$

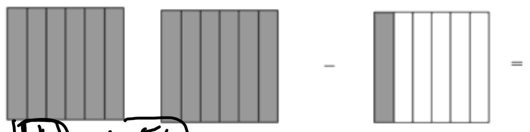
b) 2 radians $\frac{2 \times 180^\circ}{\pi} = 114.59^\circ$

Don't forget the $^\circ$

Working with Radians in Fraction Form

Because π radians is the size of a straight angle (half a rotation), we end up working a lot with angles written as fractional parts of π . Let's review adding/subtracting fractions.

$2-\frac{1}{6}=$



$\frac{12}{6}-\frac{1}{6}=\boxed{\frac{11}{6}}=\boxed{1\frac{5}{6}}$

Try

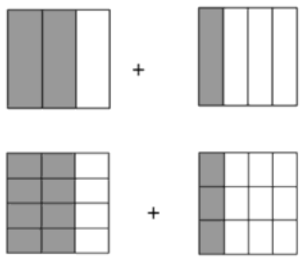
a) $1+\frac{1}{4}=$ b) $2-\frac{1}{3}=$ c) $1-\frac{1}{6}=$

$\frac{4}{4}+\frac{1}{4}=\frac{5}{4}$

d) $2-\frac{1}{4}=$ e) $1-\frac{1}{4}=$ f) $1+\frac{1}{3}=$

Different Denominators

$\frac{2}{3}+\frac{1}{4}=$



Try

a) $\frac{1}{2}+\frac{1}{3}=$ b) $\frac{7}{6}+\frac{1}{4}=$ c) $\frac{3}{5}+\frac{1}{7}=$

Is π in the fraction? It still works the same way!

Try

a) $\pi+\frac{\pi}{6}=$ b) $2\pi-\frac{\pi}{2}=$ c) $\frac{3\pi}{4}+2\pi=$

$\frac{6\pi}{6}+\frac{1\pi}{6}=\frac{7\pi}{6}$

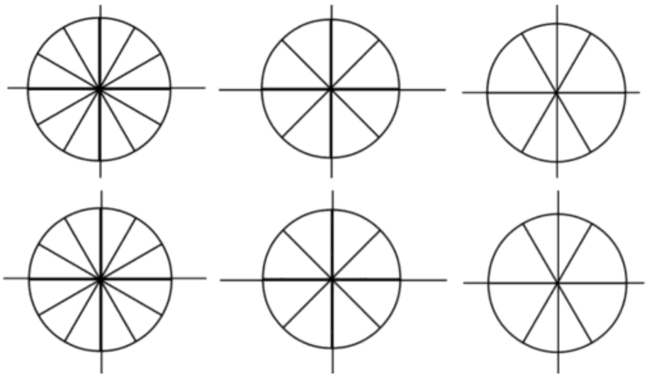
$\frac{4\pi}{2}-\frac{\pi}{2}=\frac{3\pi}{2}$

$\frac{3\pi}{4}+\frac{8\pi}{4}$

$=\frac{11\pi}{4}$

Standard-Position Angles in Radian Measure

A straight angle measures π radians. This helps us when sketching angles that are fractions involving π.



$\frac{6\pi}{6} = \frac{1}{2}$ circle $= \pi$

For each angle below:

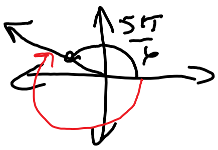
- Draw the angle in standard position.
- List two angles that are coterminal to the given angle. Use radians, not degrees!

2π

a) $\frac{5\pi}{6}$ $\quad \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$

$\frac{5\pi}{6} + -2\pi$

$\frac{5\pi}{6} + -\frac{12\pi}{6} = -\frac{2\pi}{6}$

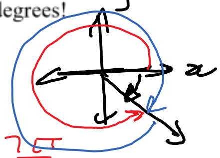


b) $-\frac{\pi}{4} + 2\pi$

$-\frac{\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$

$-\frac{\pi}{4} + -2\pi = -\frac{\pi}{4} + -\frac{8\pi}{4}$

$= -\frac{9\pi}{4}$



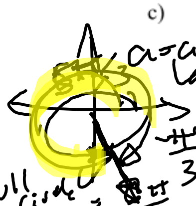
$\pi = \frac{1}{2}$ circle.

c) $\frac{5\pi}{3} + -2\pi$

$\frac{5\pi}{3} + -\frac{6\pi}{3}$

$= -\frac{\pi}{3}$

$\frac{5\pi}{3} + \frac{6\pi}{3} = \frac{11\pi}{3}$



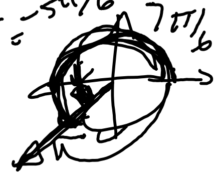
$\frac{5\pi}{3}$ a = arc length $\quad -\frac{\pi}{3}$

2π = full circle $= \frac{6\pi}{3}$

arc length.
$a = r \times \theta$
radius, in radians

d) $\frac{7\pi}{6} + -2\pi$

$\frac{7\pi}{6} + -\frac{12\pi}{6} = -\frac{5\pi}{6}$ $\quad \frac{7\pi}{6}$



$\frac{7\pi}{6} + 2\pi$

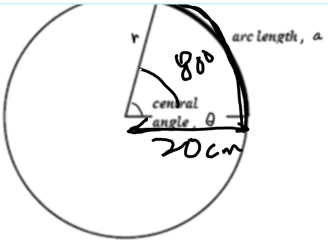
$2\pi = \frac{12\pi}{6}$

$\frac{7\pi}{6} + \frac{12\pi}{6} = \boxed{\frac{19\pi}{6}}$

Arc Length

Suppose we have a circle with radius = 20 cm. If we mark off a central angle measuring 80°, what arc length (length along the circle's circumference) does the angle cut off?

$a = r\theta$
in radians!!



in radians!!

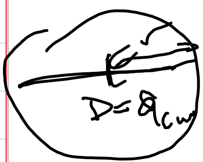
$$\theta = \frac{80° \times \pi}{180°}$$

$$a = (\cancel{20}^{1}\,cm)\left(\frac{80°\,\pi}{\cancel{180°}_{9}}\right)$$

$$\doteq \underline{27.93\,cm} \leftarrow \text{units, too!}$$

to 2 dec. places!

Suppose we have another circle, this one with diameter 9 cm. If we know that an arc measuring 7 cm is subtended by a central angle, what is the measure of that central angle, in radians? What is the measure of the central angle, in degrees?



$a = 7\,cm$

$\theta = ?$

$r = 4.5\,cm$ $(\frac{1}{2} \times D)$

$$\theta = \frac{a}{r} \qquad a = r\theta \qquad \theta = \frac{a}{r}$$

$$\theta = \frac{7}{4.5}$$

$$= 1.5555\ldots$$

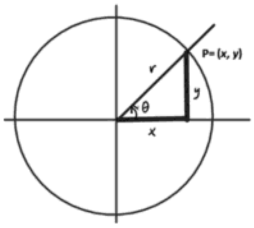
$$\theta = \underline{1.56}$$

$$\theta = 1.5555\ldots \times \frac{180°}{\pi}$$

$$= 89.1267...\ldots = \underline{89.13°}.$$

4.2 The Unit Circle

A circle is the set of all points that are a certain distance, ***radius***, from a given point, the center. Using the Pythagorean Theorem, we can get an equation for a circle.

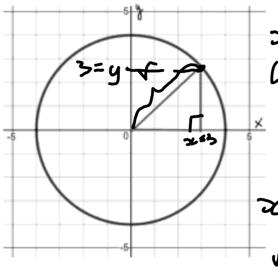


The equation for a circle with center (0,0) and radius ***r*** is:

$$x^2 + y^2 = r^2$$

Try

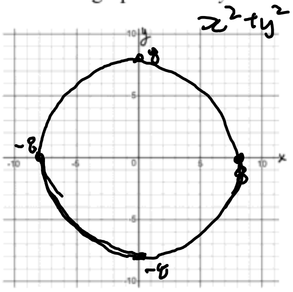
a) Find the equation of this circle.



$x^2+y^2=r^2$
$(3)^2+(3)^2=r^2$
$9+9=r^2$
$r^2=18$
$x^2+y^2=18$
$r=\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\cdot\sqrt{2}$
$=3\sqrt{2}$

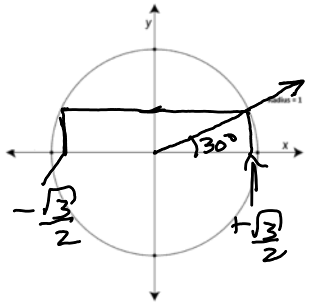
b) Sketch the graph of $x^2 + y^2 = 64$

$x^2+y^2=r^2=64$
$r=\sqrt{64}$
$=8$



Unit Circle

If we choose $r = 1$, we get a circle with radius 1 unit in length. This is called the ***unit circle***, and its equation is $\boxed{x^2 + y^2 = 1}$.



a) Is the point (0.6, 0.4) on the unit circle?

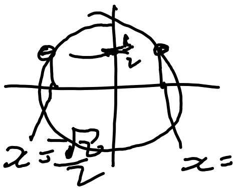
$(0.6)^2+(0.4)^2 \stackrel{?}{=} 1$
$0.52 \neq 1$
NO

b) The point below is on the unit circle. Use the unit circle equation, $x^2 + y^2 = 1$, to find the value of the unknown coordinate.

$\left(x, \frac{1}{2}\right)$
x, y

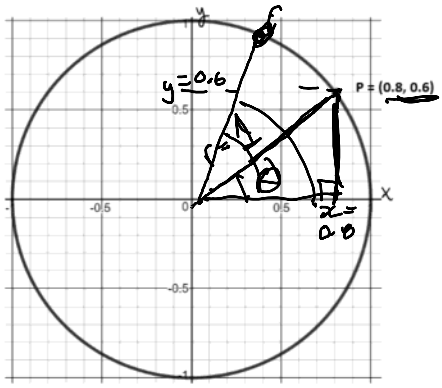
$x^2+\left(\frac{1}{2}\right)^2=1$
$x^2+\frac{1}{4}=1$
$x^2=1-\frac{1}{4}$
$x^2=\frac{3}{4}$

$1=\frac{4}{4}$
$1-\frac{1}{4}=\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$



$x=\frac{\sqrt{3}}{2}$ $\quad x=+\frac{\sqrt{3}}{2}$ $\quad x=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{\sqrt{4}}=\frac{\pm\sqrt{3}}{2}$

On the unit circle below, we have a point P, with coordinates (0.8, 0.6). We draw a line segment connecting P to the origin, (0, 0). This radius and the x-axis form a standard position angle, which we call θ. Because this is an accurate drawing, we could use a protractor and get the size of angle θ – it is about 36.87°. By drawing in a line segment that connects P to the x-axis, we create a right-triangle, with the right-angle on the x-axis.



From the diagram, we get:

$\cos\theta = \frac{adj}{hyp} = \frac{0.8}{1} = 0.8 = x$ $\quad \theta = \cos^{-1}(0.8) \doteq 36.87^\circ$

$\sin\theta = \frac{opp}{hyp} = \frac{0.6}{1} = 0.6 = y$ $\quad \theta = \sin^{-1}(0.6) \doteq 36.87^\circ$

Using the calculator, we get:

$\cos 36.87^\circ \doteq 0.8$

$\sin 36.87^\circ \doteq 0.6$

> Let $P(\theta) = (x, y)$ be the point where the terminal arm of a standard-position angle θ intersects the **unit circle**. Then we know:
> - the x-coordinate's value is equal to the cosine of the angle $\quad x = \cos\theta$
> - the y-coordinate's value is equal to the sine of the angle $\quad y = \sin\theta$

We now have a way to find sine and cosine values for ANY angle, including:

- negative angles
- 0°
- angles larger than 90°

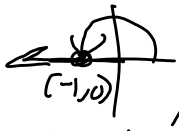
Using the triangle definitions (SOHCAHTOA) for those types of angles doesn't really make sense. For example, what would be the adjacent, opposite, and hypotenuse lengths for an angle measuring 0°?

Try – NO calculator. *(You don't need it! You can figure them out yourself!)*

$P(0^\circ) = (1, 0)$ $\quad \cos(0^\circ) = 1$ $\quad \sin(0^\circ) = 0$

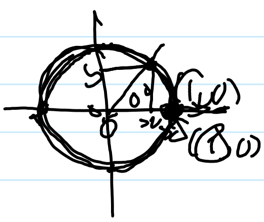
$P\left(\frac{3\pi}{2}\right) = (0, -1)$ (x y) $\quad \sin\left(\frac{3\pi}{2}\right) = y = -1$ $\quad \cos\left(\frac{3\pi}{2}\right) = 0$

$\left.\begin{matrix}P(180^\circ)\\P(\pi)\end{matrix}\right\} = (-1, 0)$



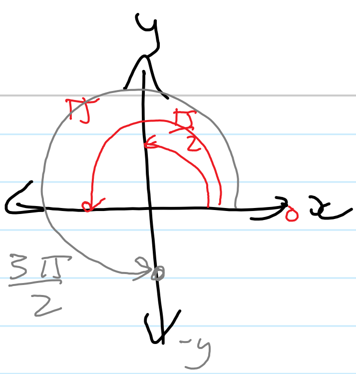
$\sin 180 = \sin\pi = 0$

$\cos 180^\circ = \cos\pi = -1$

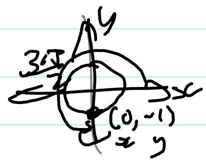


$\cos(0^\circ) = x = 1$

$\sin(0^\circ) = y = 0$



$\theta = \frac{3\pi}{2}$



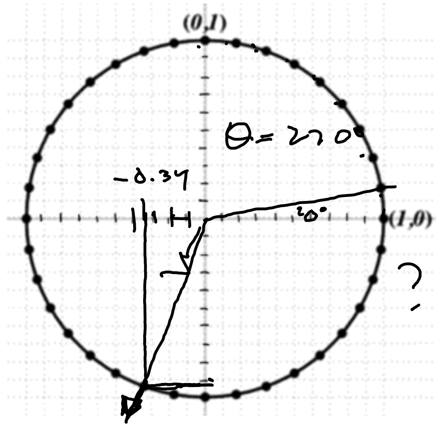
Finding Approximate Values of Trigonometric Ratios

Estimate each value using the graph at right. Compare with the calculator answer, correct to 4 decimal places.

a) $\cos 250^\circ =$ −0.34 $\sin 250^\circ =$ −0.92

b) $\cos 500^\circ =$ $\sin 500^\circ =$

c) $\cos(-10^\circ) =$ $\sin(-10^\circ) =$



http://www.malinc.se/math/trigonometry/unitcircleen.php

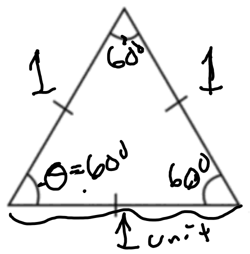
Special Triangle Angles

Besides the quadrantal angles, there are some other angles for which we can find ***exact*** coordinates for $P(\theta)$. These angles relate to special triangles.

Remember special triangles?

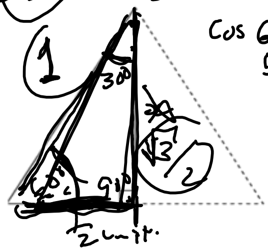
60° or $\frac{\pi}{3}$

$60^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{3}$



3 equal sides → equilateral triangle.
3 equal angles

$\sin 60^\circ = \sqrt{3}/2$
$\cos 30^\circ = \sqrt{3}/2$



$\sin 30^\circ = \frac{1/2}{1} = \frac{1}{2}$
$\pi/6$

$\cos 60^\circ = \frac{1/2}{1} = \frac{1}{2}$
$\frac{\pi}{3}$

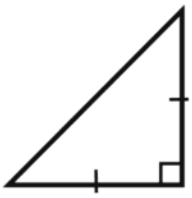
$\left(\frac{1}{2}\right)^2 + x^2 = 1^2$

$\frac{1}{4} + x^2 = 1$

$x^2 = 1 - \frac{1}{4} = 3/4$

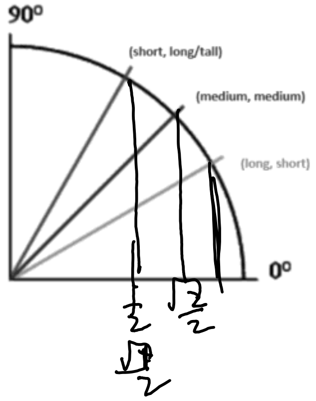
$x = \sqrt{3/4}$

$= \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$



Let's use those triangle angles in the unit circle setting. We need to adjust the size of the triangles, making their hypotenuse length = 1.

Same triangle – just turned on its side.



Short side length = $\frac{\sqrt{1}}{2} = \frac{1}{2}$

Medium side length = $\frac{\sqrt{2}}{2}$

Tall/long side length = $\frac{\sqrt{3}}{2}$

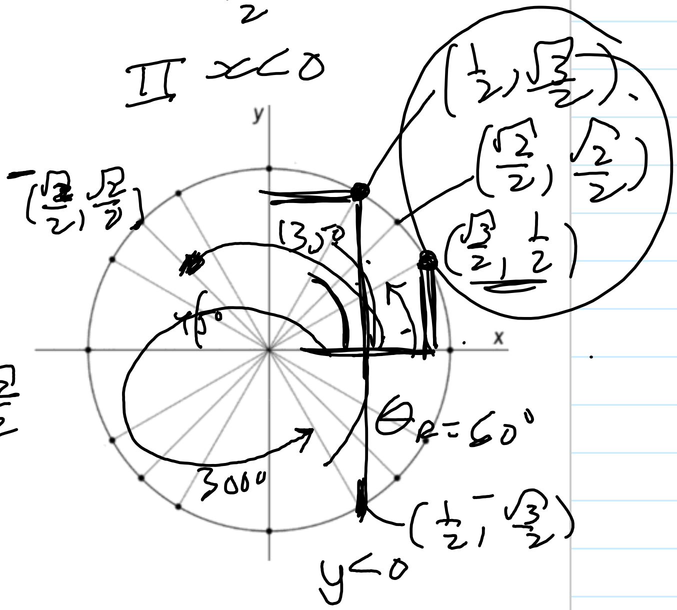
We know that the shortest side of a triangle is across from its smallest angle. This helps us label the coordinates correctly for different angles.

Try – NO calculator. Get the ***exact values***.

$\cos(30^\circ) = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin(30^\circ) = \frac{1}{2}$

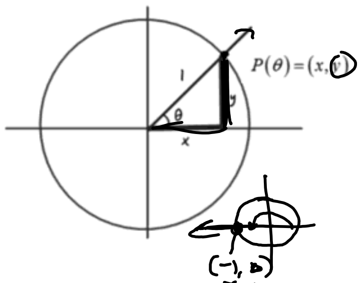
$\sin(135^\circ) = +\frac{\sqrt{2}}{2}$ $\qquad$ $\cos(135^\circ) = -\frac{\sqrt{2}}{2}$

$\cos(300^\circ) = \frac{1}{2}$ $\qquad$ $\sin(300^\circ) = \frac{\sqrt{3}}{2}$



4.3 Trigonometric Ratios

We can use the unit circle diagram to get definitions for all six of the trigonometric ratios.



Primary Ratios	***Reciprocal Ratios***
$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

More Exact Values – Use the unit circle to find each ***exact value – no calculator!***

a) $\cos(\pi) = x = -1$ $\quad\sin(\pi) = y = 0$ $\quad\tan(\pi) = \frac{y}{x} = \frac{0}{-1} = 0$

$\sec(\pi) = \frac{1}{x} = \frac{1}{-1} = -1$ $\quad\csc(\pi) = \frac{1}{y} = \frac{1}{0}$ undefined $\quad\cot(\pi) = \frac{x}{y} = \frac{-1}{0}$ undefined

$7\pi/4$ $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ x y

b) $\cos\left(\frac{7\pi}{4}\right) = x = \frac{\sqrt{2}}{2}$ $\left(\frac{1}{\sqrt{2}}\right)$ $\quad\sin\left(\frac{7\pi}{4}\right) = y = -\frac{\sqrt{2}}{2}$ $\left(-\frac{1}{\sqrt{2}}\right)$ $\quad\tan\left(\frac{7\pi}{4}\right) = \frac{y}{x} = \frac{-\sqrt{2}/2}{+\sqrt{2}/2} = -1$

$\theta_R = \frac{\pi}{4}$

$\sec\left(\frac{7\pi}{4}\right) = \frac{2}{\sqrt{2}} = \frac{2\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}$ $\quad\csc\left(\frac{7\pi}{4}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$ $\quad\cot\left(\frac{7\pi}{4}\right) = \frac{1}{-1} = -1$

$\theta_R = \frac{\pi}{6}$ y x

c) $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ $\quad\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$ $\quad\tan\left(\frac{7\pi}{6}\right) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \left(= \frac{\sqrt{3}}{3}\right)$

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ x y

$\sec\left(\frac{7\pi}{6}\right) = \left(-\frac{2}{\sqrt{3}}\right)$ same $\quad\csc\left(\frac{7\pi}{6}\right) = -\frac{2}{1} = -2$ $\quad\cot\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$

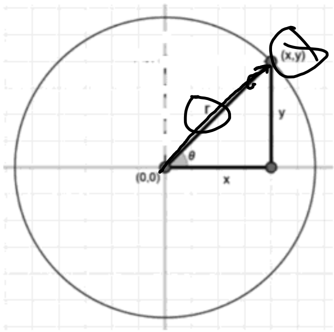
$= \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \left(\frac{-2\sqrt{3}}{3}\right)$

$\tan\left(\frac{7\pi}{6}\right) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{2} \div -\frac{\sqrt{3}}{2}$

$= -\frac{1}{2} \times -\frac{2}{\sqrt{3}} = \frac{+\cancel{2}}{-\cancel{2}\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad\left(\frac{\sqrt{3}}{3}\right)$

Sometimes the radius is not 1. Here are the ratio definitions that work for any r value.

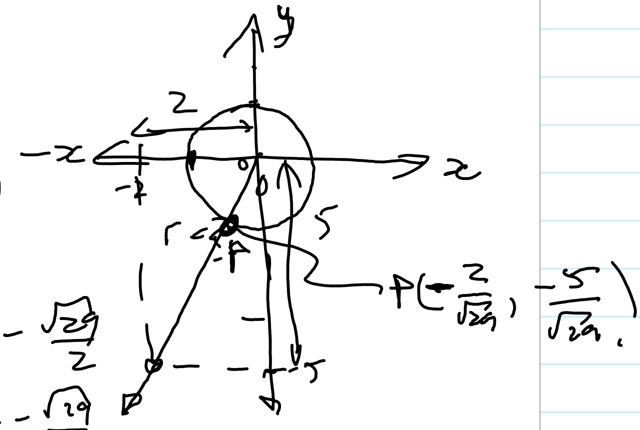


Primary Ratios	***Reciprocal Ratios***
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

Know these definitions!

Example
The terminal arm of a standard position angle θ contains the point $(-2,-5)$. Find the value of all six trigonometric ratios for angle θ. You do not need to find the size of angle θ. Leave answers in ***exact fractional form***.

The angle terminates in quadrant III.



$x =$ -2 $\quad r^2 = (-2)^2 + (-5)^2$

$y =$ -5 $\quad r^2 = 4 + 25 = 29$

$\quad r = \sqrt{29}$

$r =$ $\sqrt{29}$

$P\left(-\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right)$

$\cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{29}}$ $\qquad \sec\theta = -\frac{\sqrt{29}}{2}$

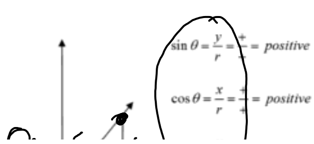
$\sin\theta = \frac{y}{r} = -\frac{5}{\sqrt{29}}$ $\qquad \csc\theta = -\frac{\sqrt{29}}{5}$

$\tan\theta = \frac{y}{x} = \frac{5}{2}$ $\qquad \cot\theta = \frac{2}{5}$

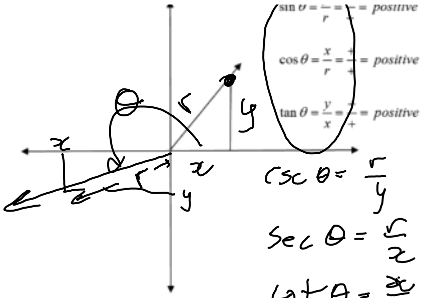
The coordinates of the point, $P(\theta)$, where the terminal arm of θ intersects the unit circle are $(\cos\theta, \sin\theta)$.
$\therefore$ P has coordinates $\left(\frac{-2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}\right)$.

Finding the Signs of the Trigonometric Ratios
How can we predict whether a specific trigonometric ratio will be positive or negative?

r - positive in every quadrant
x - depends on the quadrant, can be + or –
y - depends on the quadrant, can be + or –



x - depends on the quadrant, can be + or –
y - depends on the quadrant, can be + or –



$\csc\theta = \frac{r}{y}$

$\sec\theta = \frac{r}{x}$

$\cot\theta = \frac{x}{y}$

Example

Given the information below, decide in which quadrant (or quadrants) angle θ can terminate.

a) $\tan\theta < 0$

QII, QIV

b) $\csc > 0$

$\left(\frac{1}{\sin\theta}\right)$

QI, QII

c) $\sin\theta < 0$ *and* $\cot\theta > 0$

$\tan\theta > 0$

QIII

d) $\sec\theta > 0$ *and* $\tan\theta < 0$

$\frac{1}{\cos\theta} > 0$

$\cos\theta > 0$

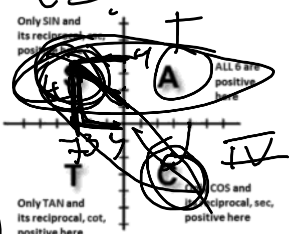
QIV

S A T C

Example

Find the value of $\sec\theta$, if we know $\tan\theta = -\frac{4}{3}$ and $\sin\theta > 0$.

$\tan\theta <, \sin\theta > 0$ QII



$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$

in Q2, $y > 0$, $x < 0$

$\sin\theta = \frac{y}{r}$

$\tan\theta = -\frac{4}{3} = \frac{y}{x} = \frac{4}{-3}$

$x = -3$

$y = 4$

$(-3, 4)$ is on the terminal arm of θ

$r^2 = (-3)^2 + (4)^2$

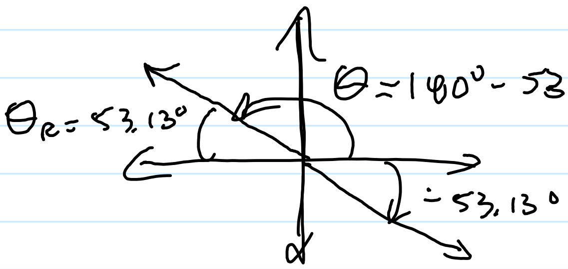
$= 25$ $\quad r = \sqrt{25} = 5$

$\cos\theta = \frac{x}{r} = -\frac{3}{5}$

$\sec\theta = \frac{5}{-3} = \boxed{-\frac{5}{3}}$

$(-1.\overline{66})$.

Check: $\theta = \tan^{-1}\left(-\frac{4}{3}\right) = -53.13^\circ$?

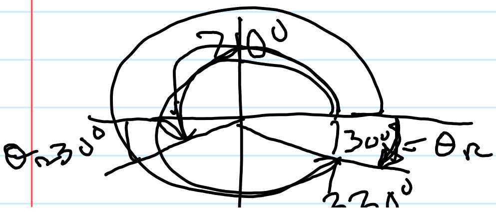


$\theta = 180^\circ - 53.13^\circ = 126.87^\circ$

$(\sec 126.87..^\circ = -1.\overline{66})$

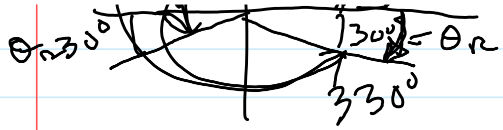
$\sin\theta = -\frac{1}{2}$ $\quad \theta = ?$

Your calculator isn't very smart.



$0 \leq \theta < 360^\circ$

It always wants to



It always wants to give you the smallest possible angle (either + or −).

Approximate Values

For angles not related to special angles, calculators can give us accurate approximations.

Try

Evaluate each of the following ratios correct to 4 decimal places, using a calculator.

a) $\tan(-65^\circ) =$ −2.1445

b) $\sec 417^\circ =$ 1.8361
cos417° = ——
then take the reciprocal
1 ÷ (cos417°)

c) $\cot\left(\frac{3\pi}{5}\right) =$ −0.3249
rad.

d) $\cos(4) =$ −0.6536
rad.

e) $\sin(-200^\circ) =$ 0.3420

f) $\csc\left(\frac{4\pi}{7}\right) =$ 1.0257
rad.
$\frac{1}{\sin(\frac{4\pi}{7})}$

Use the correct mode – either degrees or radians, matching the angle's units.

Be careful when evaluating reciprocal ratios.
Don't take the reciprocal of the angle!
Instead, use these relationships:

$$\csc(angle) = \frac{1}{\sin(angle)}$$

$$\cot(angle) = \frac{1}{\tan(angle)}$$

$$\sec(angle) = \frac{1}{\cos(angle)}$$

$\sec 26^\circ = \frac{1}{(\cos 26^\circ)}$
$\neq \cos\left(\frac{1}{26^\circ}\right)$

4.4 Solving Trigonometric Equations

We now know how to find trigonometric ratio values for any angle. Sometimes we must settle for an approximation, other times we can give an exact value.

For example:

$\sin 47^\circ =$ $\qquad\qquad$ $\cos\left(\frac{5\pi}{6}\right) =$

Now we look at the opposite situation. Given the value of a trigonometric ratio, how do we find out the size of the ***angle***? This is a process that works well:

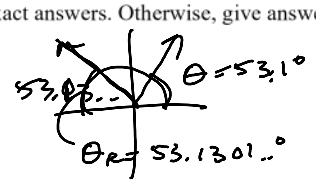
the trig function.

Isolate - Decide - Get Reference Angle - Solve

Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta - 0.8 = 0$, for $0^\circ \le \theta < 360^\circ$

+0.8 +0.8



$\sin\theta = 0.8 \leftarrow$ Q I & II.

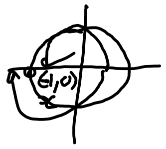
$\theta = \sin^{-1}(0.8) = 53.1301...$

$\theta = 53.1^\circ$

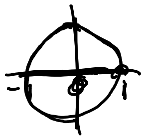
$\theta = 180^\circ - 53.1301...^\circ = 126.9^\circ$.

b) $\cos\theta = -1$, for $-\pi \le \theta < 2\pi$

$\cos\theta = -1$



$\theta = -\pi, \pi$



c) $3\cos\theta - 12 = 0$, for $0 \le \theta < 2\pi$

$+12 \quad +12$

$\frac{3\cos\theta}{3} = \frac{12}{3}$ $\cos\theta = 4$ (rad)

$\theta = \cos^{-1}(4) =$ no solution

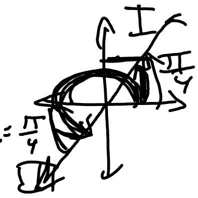
$-1 \le \cos\theta \le 1$

d) $3\tan\theta - 3 = 0$, for $0 \le \theta < 2\pi$

$+3 \quad +3$

$\frac{3\tan\theta}{3} = \frac{3}{3}$

$\tan\theta = +1$

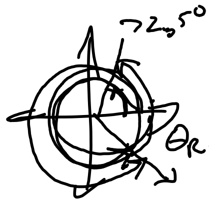


$\theta_R = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\theta \longrightarrow 720^\circ$

e) $3\sec\theta - 10 = 0$, for $0^\circ \le \theta < 720^\circ$



$\theta_R = 72.5^\circ$

$3\sec\theta = 10$

$\sec\theta = \frac{10}{3}$ + → QI, IV

$\frac{1}{\cos\theta}$ $\cos\theta = \frac{3}{10}$

In QI, $\theta = \cos^{-1}(3/10) = 72.5423..^\circ = \theta_R$
$= 72.5^\circ$

and $\theta = 72.5^\circ + 360^\circ = 432.5^\circ$

In QIV, $\theta = 360^\circ - 72.5^\circ = 287.452..$
$= 287.5^\circ$

and $\theta = 287.452..^\circ + 360^\circ$
$= 647.5^\circ$

DO NOT DO THIS, EVER:

$\cos\theta = 3/10 = 72.5^\circ$

$\cos\theta$ θ

$\cos\theta \neq \theta$

$\theta = \cos^{-1}(3/10)$ Correct

$\cos\theta = \frac{3}{10} = 72.5^\circ$

LAZY MATH

$\cos^{-1}$ Does NOT mean $\frac{1}{\cos}$

inverse function

recall $f(x)$ $f^{-1}(x)$ ch.1.

CRAZY MATH

Isolate - Decide - Get Reference Angle - Solve

1) Isolate the trigonometric term. If it uses cot, sec, or csc, take the ***reciprocal*** of both sides of the equation to get a simpler-to-solve version of the equation.

2) Decide whether the equation can be solved using

- special angles on the unit circle. Look for: $0, \pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}$ $\pm\sqrt{3}$, $\pm\sqrt{2}$
- the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ button on the calculator
- OR, cannot be solved

3) Determine in which quadrants answers will be found. look at the restriction on the domain.

4) Find the reference angle and use it to find all the solutions in the given domain. Use the same units (either degrees or radians) as shown in the question's domain. $0 \le \theta < 2\pi$ $(360°)$

Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta = -0.8$, for $0^\circ \le \theta < 360^\circ$ more next time

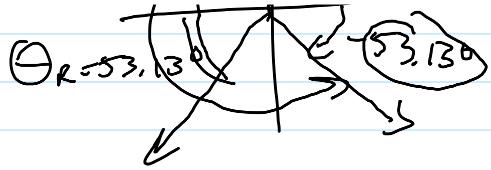
If we calculate $\sin^{-1}(-0.8)$, the calculator gives us a negative answer. We don't want this, because the domain asks for only positive answers.
To avoid this problem, give the calculator the trigonometric ratio as a positive quantity. This guarantees the calculator will give us the ***reference angle***, which will be positive.

b) $5\cos\theta - 2 = 2\cos\theta - 4$, for $0^\circ \le \theta < 720^\circ$

$\sin\theta = -0.8$ over $0 \le \theta < 360°$

$\theta = \sin^{-1}(0.8) = -53.13°$ $\theta_R = 53.13°$ $53.13°$

$\theta = \sin^{-1}(0.8) = -53.13°$



$\sin\theta$ is neg in QIII & QIV

In QIII, θ

$= 360° - 53.13° =$

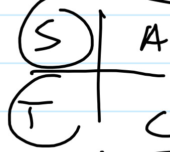
$= 180° + 53.13° = 233.13°$

In QIV, $\theta = 360° - 53.13° = \boxed{306.87°}$

---

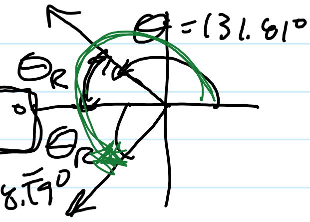
b) $5\cos\theta - \cancel{2} = 2\cos\theta - 4$ for $0 \le \theta < 720°$

$-2\cos\theta \quad +2 \quad \cancel{-2\cos\theta} \quad +2$



$3\cos\theta = -2$

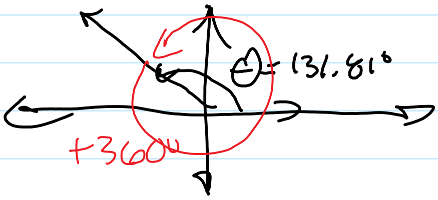
$\cos\theta = -\frac{2}{3} \quad \theta = \cos^{-1}\left(-\frac{2}{3}\right) = \boxed{131.81°}$



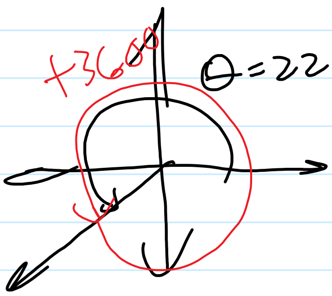
$\theta_R = 180° - 131.81° =$
$= 48.19°$

In QIII, $\theta = 180° + 48.19°$
$= \boxed{228.19°}$

Between 0° & 720°, there are 4 angles that have $\cos\theta = -\frac{2}{3}$



$\theta = 131.81° + 360° = \boxed{491.81°}$



$\theta = 228.19° + 360°$
$= \boxed{588.19°}$

General Solution

How many solutions a trigonometric equation has depends on the domain specified in the question. When the domain is all real numbers, there are infinitely many solutions. This is called the ***general solution***.

Example Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

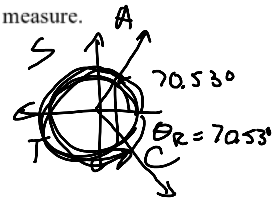
a) $3\cos\theta - 1 = 0$, ***general solution*** in degree measure.

$3\cos\theta = 1$

$\cos\theta = \frac{1}{3}$ QI

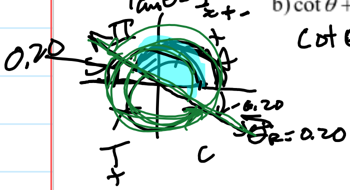
$\theta = \cos^{-1}\left(\frac{1}{3}\right) = \boxed{70.53°}$

In QIV, $\theta = 360° - 70.53$
$= \boxed{289.47°}$



$\boxed{\theta = 70.53° + 360°n,\ 289.47° + 360°n,\ n \in I}$

b) $\cot\theta + 5 = 0$, ***general solution*** in radian measure.



$\cot\theta = -5$ $\tan\theta = -\frac{1}{5}$ $\tan\theta < 0$ in Q's II & IV

$\tan^{-1}\left(-\frac{1}{5}\right) = -0.20$

In QII, $\theta = \pi - 0.20$
$= 2.94$

General solution
$\theta = 2.94 + n\pi,\ n \in I$

Here's how to write the ***general solution***:

- list each answer θ, found in one full rotation, separately
- to each answer add on the appropriate amount, either
  $+2\pi n,\ n \in I$ or $+360°n,\ n \in I$

sin cos
csc sec.

For equations using *tangent* or *cotangent*, we find that in one full rotation the two solutions are spaced exactly π or $180°$ apart. Because of this, we can just write the first solution, and add onto it:

$+\pi n,\ n \in I$ or $+180°n,\ n \in I$

Example

Suppose that for a certain equation, we are all told its solutions for $0° \le \theta < 360°$ are $\theta = 20°$ and $\theta = 160°$. What is the ***general solution***?

assuming we solved for θ using $\cos^{-1}$ or $\sin^{-1}$

$$\left.\begin{aligned}\theta &= 20° + 360°n \\ \theta &= 160° + 360°n\end{aligned}\right\} n \in I$$

Solving Second-Degree Trigonometric Equations

When we solve equations with an exponent we usually start by factoring.

For example, solve: $2x^2 - 1 = x$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$2x+1=0 \qquad x-1=0$

$\boxed{x = \tfrac{1}{2} \qquad x = 1}$

Example

Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

$$2\tan^2\theta - 1 = \tan\theta \text{, for } 0^\circ \le \theta < 360^\circ$$

$-\tan\theta \quad -\tan\theta$

let $\tan\theta = x$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$2\tan^2\theta - \tan\theta - 1 = 0$

$(2\tan\theta + 1)(\tan\theta - 1) = 0$

$2\tan\theta + 1 = 0 \qquad \tan\theta - 1 = 0$

Q II, IV $\quad \tan\theta = -\tfrac{1}{2}$

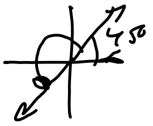
$\theta_R = \tan^{-1}(\tfrac{1}{2}) = 26.57^\circ$

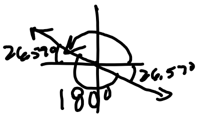
$\tan\theta = 1 \quad$ Q I, III

$\theta = \boxed{45^\circ}$

$\theta = 180^\circ + 45^\circ$

$= \boxed{225^\circ}$



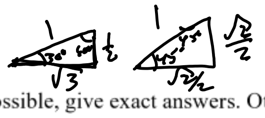


in Q II,

$\theta = 180^\circ - 26.57^\circ$

$= \boxed{153.43^\circ}$

in Q IV $\quad \theta = 360^\circ - 26.57^\circ = \boxed{333.43^\circ}$

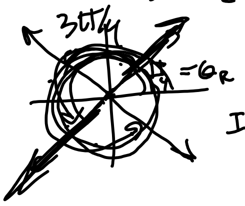


Example

Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

$2\cos^2\theta - 1 = 0$, for $0 \le \theta < 2\pi$

$\cos^2\theta = \frac{1}{2}$ $\quad \cos\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}}$



$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

In QI, $\theta = \frac{\pi}{4}$

QII, $\theta = \frac{3\pi}{4}$

QIII $\theta = \frac{5\pi}{4} \leftarrow \frac{\pi}{4} + \frac{4\pi}{4} = \frac{\pi}{4} + \pi$

QIV $\theta = \frac{7\pi}{4} \leftarrow \frac{3\pi}{4} + \frac{4\pi}{4} = \frac{3\pi}{4} + \pi$

General solutions: $\left.\begin{aligned}\theta &= \frac{\pi}{4} + \pi n\\ \theta &= \frac{3\pi}{4} + \pi n\end{aligned}\right\}\ n \in I$